The Measurement of Statistical Evidence
Lecture 5 - part 2

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Example *location normal*

- $x = (x_1, \ldots, x_n) \overset{i.i.d.}{\sim} N(\mu, \sigma_0^2)$ with $\mu \in R^1, \sigma_0^2$ known and π a $N(\mu_0, \tau_0^2)$ dist., recall we discussed eliciting $(\mu_0, \tau_0^2))$ and derived posterior

$$\begin{aligned} \mu \,|\, x &\sim N\left(\mu_x, \tau_x^2\right) \\ \mu_x &= \tau_x^2 \left(\frac{\mu_0}{\tau_0^2} + \frac{n\bar{x}}{\sigma_0^2}\right),\ \tau_x^2 = \left(\frac{1}{\tau_0^2} + \frac{n}{\sigma_0^2}\right)^{-1} \end{aligned}$$

- since $\bar{x}$ is a mss $RB(\mu \,|\, x) = RB(\mu \,|\, \bar{x})$

$$RB(\mu \,|\, \bar{x}) = \frac{f_{\mu,\bar{X}}(\bar{x})}{m_{\bar{X}}(\bar{x})} = \frac{(n/2\pi\sigma_0^2)^{1/2} \exp\{-n(\bar{x}-\mu)^2/2\sigma_0^2\}}{m_{\bar{X}}(\bar{x})}$$

where with $Z_1, Z_2 \overset{i.i.d.}{\sim} N(0,1)$ the prior predictive dist. of $\bar{X}$

$$\begin{aligned} \bar{X} &= \mu + \frac{\sigma_0}{\sqrt{n}} Z_1 = \mu_0 + \tau_0 Z_2 + \frac{\sigma_0}{\sqrt{n}} Z_1 \sim N(\mu_0, \tau_0^2 + \sigma_0^2/n) \\ m_{\bar{X}}(\bar{x}) &= (2\pi(\tau_0^2 + \sigma_0^2/n))^{-1/2} \exp\{-(\bar{x}-\mu_0)^2/2(\tau_0^2 + \sigma_0^2/n)\} \end{aligned}$$

- so the relative belief estimate is $\mu(x) = \bar{x} \to \mu_{true}$ as $n \to \infty$ and

$$\begin{aligned} PI(x) &= \{\mu : (n/2\pi\sigma_0^2)^{1/2} \exp\{-n(\bar{x}-\mu)^2/2\sigma_0^2\} > m_{\bar{X}}(\bar{x})\} \\ &= \{\mu : n(\bar{x}-\mu)^2/2\sigma_0^2 < -\log\left[(n/2\pi\sigma_0^2)^{-1/2} m_{\bar{X}}(\bar{x})\right] \\ &= \bar{x} \pm \sqrt{\frac{2\sigma_0^2}{n}\log\left[\frac{(n/2\pi\sigma_0^2)^{1/2}}{m_{\bar{X}}(\bar{x})}\right]} = \bar{x} \pm c(\bar{x}) \end{aligned}$$

- so for accuracy assessment quote half-length $c(\bar{x})$ and posterior content

$$\Pi(PI(x)\,|\,\bar{x}) = \Phi\left(\frac{\bar{x}+c(\bar{x})-\mu_x}{\tau_x}\right) - \Phi\left(\frac{\bar{x}-c(\bar{x})-\mu_x}{\tau_x}\right)$$

- note

$$\begin{aligned} c^2(\bar{x}) &= \frac{2\sigma_0^2}{n}\left\{\frac{1}{2}\log\frac{n}{2\pi\sigma_0^2} - \log m_{\bar{X}}(\bar{x})\right\} \\ &= \frac{\sigma_0^2}{n}\left\{\log\frac{\tau_0^2+\sigma_0^2/n}{\sigma_0^2/n} + \frac{(\bar{x}-\mu_0)^2}{\tau_0^2+\sigma_0^2/n}\right\} \end{aligned}$$

and so $PI(x) \neq \phi$ and $PI(x) \to \{\mu_{true}\}$

- for any $\gamma \leq \Pi(Pl(x) \,|\, \bar{x})$ the γ-relative belief region $C_\gamma(x) = \bar{x} \pm k_\gamma(\bar{x})$ can be quoted where $k_\gamma(\bar{x})$ satisfies

$$\Phi\left(\frac{\bar{x} + k_\gamma(\bar{x}) - \mu_x}{\tau_x}\right) - \Phi\left(\frac{\bar{x} - k_\gamma(\bar{x}) - \mu_x}{\tau_x}\right) = \gamma$$

which can be obtained via simple tabulation

- to assess $H_0 := \{\mu_*\}$ compute $RB(\mu_* \,|\, \bar{x})$ and the strength

$$\begin{aligned} & \Pi(RB(\mu \,|\, \bar{x}) \leq RB(\mu_* \,|\, \bar{x}) \,|\, \bar{x}) \,|\, \bar{x}) \\ = \; & \Pi\left((\bar{x} - \mu)^2 \geq -\frac{2\sigma_0^2}{n} \log[(n/2\pi\sigma_0^2)^{-1/2} m_{\bar{X}}(\bar{x}) RB(\mu_* \,|\, \bar{x}) \,|\, \bar{x})] \,\middle|\, \bar{x} \right) \\ = \; & \Pi\left((\bar{x} - \mu)^2 \geq d^2(\mu_*) \,\middle|\, \bar{x} \right) \\ = \; & \Pi\left((-\infty, \bar{x} - d(\mu_*)) \cup (\bar{x} + d(\mu_*), \infty) \,\middle|\, \bar{x} \right) \\ = \; & \Phi\left(\frac{\bar{x} - d(\mu_*) - \mu_x}{\tau_x}\right) + 1 - \Phi\left(\frac{\bar{x} + d(\mu_*) - \mu_x}{\tau_x}\right) \end{aligned}$$

- while exact formulas for these quantities can be convenient these are generally not available and we need to proceed numerically

- so we do this here for this problem with a specific example (discussed previously)

- the elicitation (based on the mean lying in $(l, u) = (3, 10)$ with prob. 0.99) resulted in $(\mu_0, \tau_0) = (6.5, 1.36)$

- if $\sigma_0^2 = 2$ and $n = 10$, $\bar{x} = 7.3$ is observed, then the posterior of μ is $N\left(\mu_x, \tau_x^2\right) = N(7.23, 0.18)$

- suppose interest is in $\psi = \Psi(\mu) = \mu^3$ and a meaningful difference from μ_{true}^3 is $\delta = 10$

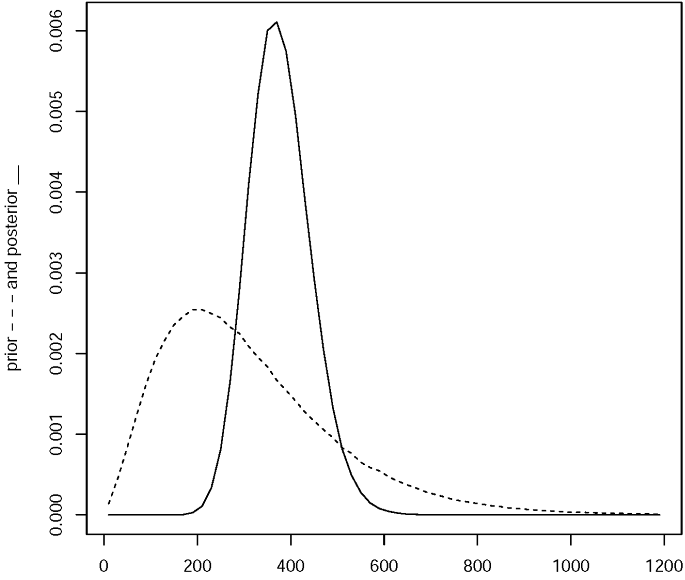
- so create grid of 60 points $grid = \{10, 30, 50, \ldots, 1190\}$ and for $i \in grid$ we estimate the prior and posterior contents of $i \pm 10$ by simulating from the prior and posterior of μ^3 and estimate the densities over these intervals by dividing these estimates by 2δ giving us density histograms

Program in R

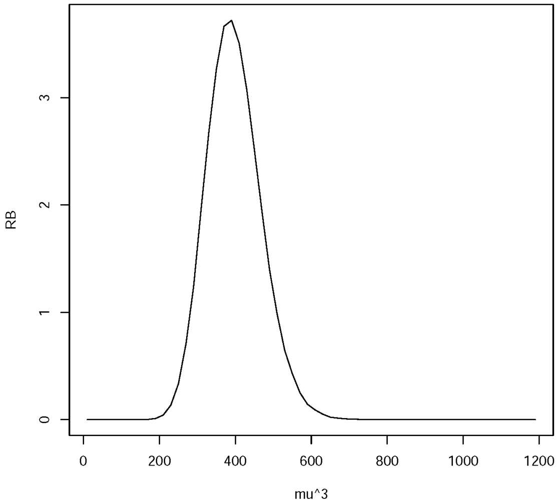
```
# set up
sigma0=sqrt(2)
mu0=6.5
tau0=sqrt(1.36)
n=10
xbar=7.3
taux2=1/(1/tau0**2 + n/sigma0**2)
mux=taux2*(mu0/tau0**2 +n*xbar/sigma0**2)
mux
taux2
taux=sqrt(taux2)
numgrid=60
grid=-10+20*c(1:numgrid)
```

```
# prior calculations
priorprob=0*grid
nmonte=500000
mugen=rnorm(nmonte,mu0,tau0)
mugen3=mugen**3
for (i in 1:numgrid){
for (j in 1:nmonte) {
if (mugen3[j]> grid[i]-10 & mugen3[j] <= grid[i]+10 ){
priorprob[i]=priorprob[i]+1
}
}
}
priorprob=priorprob/nmonte
priordens=priorprob/20
plot(grid,priordens,xlab="mu cubed", ylab="prior density", type="l",lty=2)
```

```
# posterior calculations
postprob=0*grid
nmonte=500000
mugen=rnorm(nmonte,mux,taux)
mugen3=mugen**3
for (i in 1:numgrid){
for (j in 1:nmonte) {
if (mugen3[j]> grid[i]-10 & mugen3[j] <= grid[i]+10 ){
postprob[i]=postprob[i]+1
}
}
}
postprob=postprob/nmonte
postdens=postprob/20
plot(grid,postdens,xlab="mu^3", ylab=expression("prior - - - and posterior
__"), type="l",lty=1)
lines(grid,priordens,type="l",lty=2)
```


prior - - - and posterior __
0.000
0.001
0.002
0.003
0.004
0.005
0.006
0
200
400
600
800
1000
1200

```
# compute RB
RB=postdens/priordens
plot(grid,RB,xlab="mu^3", ylab="RB", type="l",lty=1)
```



```
# find mu^3(x)
imax=1
RBmax=RB[imax]
for (i in 1:numgrid){
if( RB[i]>RBmax){
imax=i
RBmax=RB[i]}
}
cat("mu^3(x)= ",grid[imax],"RB(mu^3(x) | x) = ",RB[imax],"\n")
mu^3(x)= 390 RB(mu^3(x) | x) = 3.719956
```

- so $\mu^3(x) = 390$, $RB(390|x) = 3.719956$ and note $\bar{x}^3 = 7.3^3 = 389.017$ so good approximation

```
# approximate Pl(x) and its posterior content
plauscont=0
for (i in 1:numgrid){
if( RB[i]>1){
cat(grid[i]," ")
plauscont=plauscont+postprob[i]}
}
plauscont
290 310 330 350 370 390 410 430 450 470 490
[1] 0.903396
```

- so $PL(x) = (280, 500)$ with posterior content 0.903396

- to get a 0.80-relative belief region

```
k=2
cont=0
for (i in 1:numgrid){
if( RB[i]>k){
cat(grid[i]," ")
cont=cont+postprob[i] }
}
cont
330 350 370 390 410 430 450 > cont
[1] 0.697656
```

- too low

```
k=1.5
cont=0
for (i in 1:numgrid){
if( RB[i]>k){
cat(grid[i]," ")
cont=cont+postprob[i]}
}
cont
310 330 350 370 390 410 430 450 470 > cont
[1] 0.820932
```

- good enough so $C_{0.82}(x) = (290, 470)$

- assess $H_0 : \mu^3 = 625$ or equivalently, since $625 \in (630 - 10, 630 + 10)$ where 630 is a grid point $H_0 = (620, 640)$ and $RB(630 \,|\, x)$

```
# assess H_0: mu^3=625 by getting RB at nearest gridpoint, here 630
for (i in 1:numgrid){
cat(i,grid[i],RB[i],"\n")
}
```

- obtain $RB(630 \,|\, x) = 0.05116493$ so evidence against and strength is

```
#strength
strength=0
for (i in 1:numgrid){
if (RB[i]<=RB[32]){
strength=strength+postprob[i]
}
}
strength
[1] 0.003552
```

- so strong evidence against H_0

Example *Fieller (1954) Some problems in interval estimation. JRSSB, 16, 2, 175–185.*

- $x = (x_1, \ldots, x_m) \overset{i.i.d.}{\sim} N(\mu, \sigma_0^2)$ ind. of $y = (y_1, \ldots, y_n) \overset{i.i.d.}{\sim} N(\nu, \sigma_0^2)$ and $\psi = \Psi(\mu, \nu) = \mu/\nu$

- give an elicitation algorithm for μ and ν that takes into account that we know something about ψ

- it is probably okay to allow 0 as a possible (probable) value for μ but not both μ and ν as that would suggest the possibility that ψ is not defined

- need to choose a relevant δ

- then proceed as we did in the previous example for problems **E** and **H**

- maybe take $m = n = 10, \sigma_0^2 = 1$ and suppose $\mu_{true} = 2\nu_{true}$ where $\nu_{true} = 10$ and generate the samples x and y